# A NEW ORDER THEORY OF SET SYSTEMS AND BETTER QUASI-ORDERINGS

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ABSTRACT. By reformulating a learning process of a set system $L$ as a game between Teacher (presenter of data) and Learner (updater of the abstract independent set of the data), we define the order type $\dim L$ of $L$ to be the order type of the game tree. The theory of this new order type and continuous, monotone function between set systems corresponds to the theory of well quasi-orderings (WQOs). As Nash-Williams developed the theory of WQOs to the theory of better quasi-orderings (BQOs), we introduce a set system that has order type and corresponds to a BQO. We prove that the class of set systems corresponding to BQOs is closed by any monotone function. In (Shinohara and Arimura. "Inductive inference of unbounded unions of pattern languages from positive data." *Theoretical Computer Science*, pp. 191–209, 2000), for any set system $L$, they considered the class of arbitrary (finite) unions of members of $L$. From viewpoint of WQOs and BQOs, we characterize the set systems $L$ such that the class of arbitrary (finite) unions of members of $L$ has order type. The characterization shows that the order structure of the set system $L$ with respect to the set-inclusion is not important for the resulting set system having order type. We point out continuous, monotone function of set systems is similar to positive reduction to Jockusch-Owings' weakly semirecursive sets. Keyword: better elasticity, continuous deformation, powerset orderings, linearization, unbounded unions, wqo

## 1. Introduction

A *set system* $\mathcal{L}$ over a set $T$, a subfamily of the powerset $P(T)$, is a topic of (extremal) combinatorics [6][22], as well as a target of an algorithm to learn in computational learning theory [27].

A *well quasi-ordering* [17] (WQO for short) is, by definition, a quasi-ordering $(X, \preceq)$ which has neither an infinite antichain nor an infinite descending chain. WQOs are employed in algebra [17], combinatorics [24][36], formal language theory [7][8][15][33], and so on.

WQOs and related theorems such as Higman's theorem [17], König's lemma and Ramsey's theorem [6] are sometimes employed in computational learning theory. In [23][30][40], sufficient conditions for set systems being learnable is studied with König's lemma and Ramsey's theorem, In [39], for a set system $\mathcal{L}$, Shinohara-Arimura considered the *unbounded unions* of $\mathcal{L}$, that is, the class $\mathcal{L}^{<\omega}$ of nonempty finite unions of members of $\mathcal{L}$, and then they used Higman's theorem to study a sufficient condition for it being learnable. In [10], de Brecht employed WQOs to calibrate *mind change* complexity of unbounded unions of restricted pattern languages. Motivated by [23][30][40], a somehow systematic study on the relation between WQOs and a class of learnable set systems is done in [3], as follows:

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(i) By reformulating a learning process of a set system $\mathcal{L}$ as a game between Teacher (presenter of data) and Learner (updater of abstract independent set), we define the order type $\dim \mathcal{L}$ of $\mathcal{L}$ to be the order type of the game tree, if the tree is well-founded. According to computational learning theory, if an indexed family $\mathcal{L}$ of recursive languages has well-defined $\dim \mathcal{L}$ then $\mathcal{L}$ is learnable by an algorithm from positive data. If a set system has the well-defined order type, then we call it a *finitely elastic set system* (FESS for short). See Definition 12.

(ii) For each *quasi-ordering* $\mathcal{X} = (X, \preceq)$, we consider the set system $\mathrm{ss}\,(\mathcal{X})$ consisting of upper-closed subsets of $X$. The set system has the order type equal to the *maximal order type* [13] of $\mathcal{X}$. Furthermore, the construction $\mathrm{ss}\,(\bullet)$ has an left-inverse $\mathrm{qo}\,(\cdot)$. Here for a set system $\mathcal{L}$, $\mathrm{qo}\,(\mathcal{L})$ is a quasi-ordering $(\bigcup \mathcal{L},\ \preceq_{\mathcal{L}})$ such that

$$x \preceq_{\mathcal{L}} y \iff \forall L \in \mathcal{L}, (x \in L \implies y \in L)\,.$$

The maximal order type $\mathrm{otp}(\mathcal{X})$ of $\mathcal{X}$ is defined if and only if $\mathcal{X}$ is a WQO. For any quasi-ordering $\mathcal{X}$, if one of $\mathrm{otp}(\mathcal{X})$ and $\dim \mathrm{ss}\,(\mathcal{X})$ is defined then the other side is defined with the same ordinal number. So FESSs correspond to WQOs.

(iii) For every nonempty set $U$, the product topological space $\{0,1\}^U$ of the discrete topology $\{0,1\}$ is called a *Cantor space*. A subspace of a Cantor space is represented by $\mathcal{L}, \mathcal{M}, \ldots$. We say a function from $\mathcal{M}$ to $\mathcal{L}$ is *continuous*, if it is continuous with respect to the subspaces $\mathcal{M}, \mathcal{L}$ of the Cantor spaces. We identify $\{0,1\}^U$ with the powerset $P(U)$, and a function from $\{0,1\}^U$ to $\{0,1\}^U$ with a function from $P(U)$ to $P(U)$. We say $\mathfrak{D} : \mathcal{M} \to \mathcal{L}$ is a *deformation*, if it is monotone (i.e. $M \subseteq M'$ implies $\mathfrak{D}(M) \subseteq \mathfrak{D}(M')$.) If a deformation is continuous, then it has following finiteness condition:

**Lemma 1.** *Let* $\mathfrak{D} : \{0,1\}^{\bigcup \mathcal{M}} \to \mathcal{L}$.

(1) $\mathfrak{D}$ *is a deformation, if and only if there is a binary relation* $R \subseteq (\bigcup \mathcal{L}) \times P\,(\bigcup \mathcal{M})$ *such that*

$$\forall M \in \mathcal{M}\, \forall x \in \bigcup \mathcal{L} \quad \big(\mathfrak{D}(M) \ni x \ \Leftrightarrow\ \exists v \subseteq M.\ R(x,v)\big). \tag{1}$$

(2) $\mathfrak{D}$ *is a* continuous *deformation, if and only if there exists* $R \subseteq (\bigcup \mathcal{L}) \times P\,(\bigcup \mathcal{M})$ *such that* (1) *holds, but* $v$ *is a* finite *set whenever* $R(x,v)$ *holds, and there are only* finitely *many such* $v$*'s for each* $x$*.* ([3])

*For each binary relation* $R \subseteq (\bigcup \mathcal{L}) \times P(\bigcup \mathcal{M})$*, the function* $\mathfrak{D}$ *satisfying* (1) *is unique. So we write it by* $\mathfrak{D}_R$*. Conversely, every deformation* $\mathfrak{D} : \{0,1\}^{\bigcup \mathcal{M}} \to \mathcal{L}$ *is written as* $\mathfrak{D}_R$ *by a binary relation*

$$R := \left\{(x,v) \in \left(\bigcup \mathcal{L}\right) \times P\left(\bigcup \mathcal{M}\right)\ ;\ \mathfrak{D}(v)(x) = 1\right\}.$$

The class of WQOs is closed under finitary operations such as Higman embedding [17] and topological minor relation [14, Sect. 1.7] between finite trees [14, Ch. 12][24]. The class of finite graphs is a WQO under the *minor relation*. Robertson-Seymour's proof of it is given in the numbers IV-VII, IX-XII and XIV-XX of their series of over 20 papers under the common title of *Graph Minors*, which has been appearing in the *Journal of Combinatorial Theory, Series B*, since 1983. For a shorter proof, see recent papers by Kawarabayashi and his coauthors.

The class of FESSs enjoys a useful closure condition:

**Proposition 1** ([3])**.** *For any set systems $\mathcal{L}$ and $\mathcal{M}$ and any continuous deformation $\mathfrak{D} : \{0,1\}^{\bigcup \mathcal{M}} \to \mathcal{L}$, if $\mathcal{M}$ is an* FESS*, so is the image $\mathfrak{D}[\mathcal{M}]$ of $\mathcal{M}$ by $\mathfrak{D}$.*

By it, we prove that for various (nondeterministic) language operators (e.g. Kleene-closure, shuffle-product [35][38], shuffle-closure [18], (iterated) literal shuffle [5], union, product, intersection), the elementwise application of such operator to (an) FESS(s) induces an FESS.

Roughly speaking, a deformation transforms any quasi-ordering $\preceq$ to the *powerset ordering* [29] $\preceq^{\forall}_{\exists}$. It is through our correspondence $(\mathrm{ss}(\bullet), \mathrm{qo}(\cdot))$ between quasi-orderings to set systems (Section 3). Although the powerset ordering of Rado's WQO [34] is not a WQO [29, Corollary 12], the class of *better quasi-orderings* [32] (BQOs for short) is closed with respect to the powerset ordering. There are *infinitary* operations under which the class of WQOs is not closed but the class of BQOs is. So we introduce a *better elastic set system* (BESS for short), as a set system corresponding to a BQO. We show that the class of BESSs is closed under the image of any deformation (Section 3), where any deformations are infinitary in a sense of Lemma 1. By this and (i), we can develop the computational learning theory of $\omega$-languages [37].

The notion of BESS is useful in investigating a following set system

$$\mathcal{L}^{<\omega} := \{\bigcup \mathcal{M} \; ; \; \emptyset \neq \mathcal{M} \subseteq \mathcal{L}, \; \#\mathcal{M} < \infty\}. \tag{2}$$

It is studied for the learnability of language classes such as a class of regular pattern languages [39]. We characterize the set systems $\mathcal{L}$ such that $\mathcal{L}^{<\omega}$ are again FESSs, and then we prove that for every BESS $\mathcal{L}$, $\mathcal{L}^{<\omega}$ is an FESS.

We remark that another importance of set system $\mathcal{L}^{<\omega}$. We conjecture that the order type of an FESS is the supremum, actually the maximum, of the order types of the "linearizations" of the FESS. This conjecture corresponds to a proposition useful in investigating WQOs:

**Proposition 2** ([13])**.** *The order type of a* WQO *is the maximum order type of the linearizations of the* WQO*.*

A "linearization" of an FESS $\mathcal{L}$ seems to be a subfamily of $\mathcal{L}^{<\omega}$.

We hope that our study on FESSs and BESSs are useful in solving problems (e.g. decision problem of timed Petri-nets [1][2], the multiplicative exponential linear logic [12]) which are related to WQOs and BQOs but hard to solve with conventional arguments for WQOs and BQOs.

The rest of paper is organized as follows: In the next section, we recall the powerset ordering and Marcone's characterization [29] of WQOs such that the powerset orderings are again WQOs. We also review the combinatorial definition of a BQO. Then we recall that the class of BQOs is closed with respect to the powerset ordering. In Section 3, for the set system $\mathrm{ss}(\preceq)$ of upper-closed sets of a fixed quasi-ordering $\preceq$, the image by a deformation is essentially the set system of upper-closed sets of the powerset ordering of $\preceq$. Then we introduce a BESS as an FESS that corresponds to a BQO. Then we prove that the image of a BESS by any deformation is an FESS. In Section 4, we characterize the class of set systems $\mathcal{L}$ such that $\mathcal{L}^{<\omega}$ is an FESS, from viewpoint of BQO theory. We contrast our characterization with

Shinohara-Arimura's sufficient condition [39] for a set system $\mathcal{L}$ to have an FESS $\mathcal{L}^{<\omega}$.

In appendix, we propose to extend Ramsey numbers to estimate the ordinal order type of set systems, and then present miscellaneous results for computable analogue of (iii). Finally, we review the order type of a set system from [3].

## 2. Better quasi-orderings and powerset ordering

A better quasi-ordering (BQO for short), a stronger concept than a WQO, has pleasing closure properties with respect to

- embedding for transfinite sequences [32];
- topological minor relation for infinite trees [25][31]; and
- a *powerset ordering* $\preceq_\exists^\forall$ (see Definition 4) ([29, Corollary 10]).

2.1. **Combinatorial definition of BQOs.** We first recall the definition of BQOs by *barriers* [29] and then the closure of BQOs with respect to the powerset ordering $\preceq_\exists^\forall$. Please be advised to refer [28][29] for the detail.

Hereafter the first infinite ordinal $\omega$ is identified with the set of nonnegative integers. The class of subsets $X$ of $U$ such that the cardinality of $X$ is less than $\alpha$ (equal to $\alpha$, resp.) is denoted by $[U]^{<\alpha}$ ($[U]^\alpha$, resp.). A set $X \subseteq \omega$ is often identified with the sequence enumerating it in a strictly increasing order.

**Definition 1.** (1) *We say $B \subseteq [\omega]^{<\omega}$ a* barrier*, if (1) $\bigcup B$ is infinite; (2) for all $\sigma \in [\bigcup B]^\omega$ there exists $s \in B$ such that $s$ is a prefix of $\sigma$; and (3) for all $s, t \in B$, $s \not\subsetneq t$.*
(2) *For $s, t \in [\omega]^{<\omega}$, we write $s \lhd t$, if, the sequence $s$ is a prefix of the sequence $u = s \cup t$ and the sequence $t$ is a prefix of $u \setminus \{\min u\}$.*
(3) *Let $\mathrm{o.t.}(B)$ the maximal order type of $B$ with respect to the lexicographical ordering.*

Observe that

$$\mathcal{S}ingl := \{\ \{n\}\ ;\ n \in \omega\} \tag{3}$$

is a barrier. Any barrier $B$ of $\mathrm{o.t.}(B)$ being $\omega$ consists only of singletons, according to [29, p. 342].

We recall an $\alpha$-WQO and a BQO [29, Definition 3].

**Definition 2.** *Let $\alpha$ be a countable ordinal and $\preceq$ a quasi-ordering on $Q$. We say a function $f : B \to Q$ is* good *with respect to $\preceq$, if there are some $s, t \in B$ such that $s \lhd t$ and $f(s) \preceq f(t)$. Otherwise we say $f$ is* bad*. We say $\preceq$ an $\alpha$-WQO, if for every barrier with $\mathrm{o.t.}(B) \leq \alpha$ every function $f : B \to Q$ is good with respect to $\preceq$. If $\preceq$ is an $\alpha$-WQO for all countable ordinal $\alpha$, we call $\preceq$ a BQO.*

Because $\mathcal{S}ingl$ is a barrier, every BQO is a WQO. When we define an $\alpha$-WQO for a countable ordinal $\alpha$, we have only to consider only *smooth barriers* among the barriers, as [29] explains:

**Definition 3.** *By a* smooth *barrier, we mean a barrier $B$ such that for all $s, t \in B$ with $\#s < \#t$ there exists $i \leq \#s$ such that the $i$-th smallest element of $s$ is less than that of $t$.*

By an *indecomposable ordinal*, we mean $\omega^{\beta}$ such that $\beta$ is any ordinal. Recall [28, Corollary 3.5].

**Proposition 3.** *If $B$ is a smooth barrier, then the ordinal* o.t.$(B)$ *is indecomposable.*

Then we have [29, Theorem 4].

**Proposition 4.** *Let $\alpha$ be a countable ordinal and $\preceq$ a quasi-ordering on $Q$. $\preceq$ is an $\alpha$-*WQO *if and only if for every smooth barrier $B$ with* o.t.$(B) \leq \alpha$*, every map $f : B \to Q$ is* good *with respect to $\preceq$.*

**Corollary 1.** *$\preceq$ is a* BQO *if and only if it is an $\alpha$-*WQO *for all countable infinite indecomposable ordinal $\alpha$.*

We use properties of a barrier from [29, Lemma 6].

**Proposition 5.** *For a barrier $B$, let $B^2 \subseteq [\omega]^{<\omega}$ be*

$$B^2 := \{s \cup t\ ;\ s, t \in B,\ s \lhd t\}.$$

*Then*

(1) *for each $t \in B^2$ there exist unique $\pi_0(t), \pi_1(t) \in B$ such that $\pi_0(t) \lhd \pi_1(t)$ and $t = \pi_0(t) \cup \pi_1(t)$;*
(2) *if $t, t' \in B^2$ and $t \lhd t'$ then $\pi_1(t) = \pi_0(t')$;*
(3) *$B^2$ is a barrier; and*
(4) *if* o.t.$(B)$ *is indecomposable then* o.t.$(B^2) =$ o.t.$(B) \cdot \omega$.

2.2. **Powerset ordering.**

**Definition 4.** *For a quasi-ordering $(X,\ \preceq)$, we define a quasi-ordering on the powerset $P(X)$ by*

$$v \preceq^{\forall}_{\exists} v' :\iff \forall x' \in v' \exists x \in v.\ x \preceq x'.$$

It is studied for the reachability analysis of Petri nets (verification of infinite-state systems [1], Timed Petri net [1][2],. . ..)

**Proposition 6** ([19], [29, Theorem 9])**.** *If $\alpha$ is a countable infinite indecomposable ordinal and $(Q, \preceq)$ is an $(\alpha \cdot \omega)$-*WQO*, then the powerset $P(Q)$ ordered by $\preceq^{\forall}_{\exists}$ is an $\alpha$-*WQO*.*

**Lemma 2.** *For a quasi-ordering $\mathcal{X} = (Q, \preceq)$, the following are equivalent:*

(1) *(a) $\mathcal{X}$ is a* WQO*; and (b) let $F$ be any function from $[\omega]^2$ to $Q$. Then if $F(\{i, j\}) \prec F(\{i, j + 1\})$ for any $i < j < \omega$, then there are $i < j < k < \omega$ such that $F(\{i, j\}) \prec F(\{j, k\})$.*
(2) *$\mathcal{X}$ is an $\omega^2$-*WQO*.*
(3) *$\left([Q]^{<\omega},\ \preceq^{\forall}_{\exists}\right)$ is a* WQO*.*
(4) *$\left(P(Q),\ \preceq^{\forall}_{\exists}\right)$ is a* WQO*.*

*Proof.* The proof is similar to that of [29, Corollary 12], but we use Rado's characterization [34, Theorem 3] of WQOs $(Q, \preceq)$ such that the set of sequences of elements of $Q$ with the length $\omega$ is again a WQO. For any WQO $\mathcal{X} = (Q, \preceq)$ and any countable ordinal $\alpha$, let $\mathcal{X}^{\alpha}$ ($\mathcal{X}^{<\alpha}$ resp.) be the set of sequences of elements of $Q$ of length $\alpha$ (less than $\alpha$, resp.) quasi-ordered by naturally generalized Higman's embedding. The condition (1) is equivalent to $\mathcal{X}^{\omega}$ being a WQO, by [34, Theorem 3].

By Higman's theorem, it is equivalent to $(\mathcal{X}^\omega)^{<\omega} = \mathcal{X}^{<\omega^2}$ being a WQO. By [28], it is equivalent to (2). The equivalence to the other conditions follows from [29, Corollary 12].

## 3. Better elasticity — deformation as powerset ordering

Given a deformation $\mathfrak{D}$ and a WQO $\mathcal{X}$, we try to construct explicitly from $\mathcal{X}$ a suitable WQO $\mathcal{Y}$ such that $\mathfrak{D}[\mathrm{ss}(\mathcal{X})] \subseteq \mathrm{ss}(\mathcal{Y})$.

**Definition 5.** *Let $\mathcal{L}$ and $\mathcal{M}$ be set systems. Suppose $R \subseteq (\bigcup \mathcal{L}) \times P(\bigcup \mathcal{M})$ and $\mathcal{X} := (\bigcup \mathcal{M},\ \preceq)$ is a quasi-ordering. Then define a quasi-ordering $Q_R(\mathcal{X})$ by $(\bigcup \mathcal{L},\ \sqsubseteq)$ as follows: For any $x, x' \in \bigcup \mathcal{L}$, we write $x \sqsubseteq x'$, if whenever $R(x, v)$ holds, there exists $v'$ such that $R(x', v')$ and $v \preceq^{\forall}_{\exists} v'$.*

**Lemma 3.** *Let $\mathcal{L}$ and $\mathcal{M}$ be set systems. Suppose $R \subseteq (\bigcup \mathcal{L}) \times P(\bigcup \mathcal{M})$. If $\mathcal{X}$ is a quasi-ordering on $\bigcup \mathcal{M}$, then $Q_R(\mathcal{X})$ is indeed a quasi-ordering on $\bigcup \mathcal{L}$.*

*Proof.* Let $Q_R(\mathcal{X})$ be $(\bigcup \mathcal{L}, \sqsubseteq)$. It is easy to see that $x \sqsubseteq x$. From $x \sqsubseteq x'$ and $x' \sqsubseteq x''$, we derive $x \sqsubseteq x''$. Let $R(x, v)$. By $x \sqsubseteq x'$, there exists $v'$ such that $R(x', v')$ and $v \preceq^{\forall}_{\exists} v'$. By $x' \sqsubseteq x''$, there exists $v''$ such that $R(x'', v'')$ and $v' \preceq^{\forall}_{\exists} v''$. Because $\preceq^{\forall}_{\exists}$ is a quasi-ordering, we have $v \preceq^{\forall}_{\exists} v''$.

It seems difficult to replace a "quasi-ordering" with a "partial ordering" in Lemma 3. For a following theorem, see (ii) in the first section for a left-inverse $\mathrm{qo}(\cdot)$ of $\mathrm{ss}(\bullet)$, and Lemma 1 for the definition of $\mathfrak{D}_R$.

**Theorem 1.** *For any $R \subseteq (\bigcup \mathcal{L}) \times P(\bigcup \mathcal{M})$, we have $\mathfrak{D}_R[\mathcal{M}] \subseteq \mathrm{ss}(Q_R(\mathrm{qo}(\mathcal{M})))$.*

*Proof.* Let $(\bigcup \mathcal{M},\ \preceq) = \mathrm{qo}(\mathcal{M})$ and $(\bigcup \mathcal{L},\ \sqsubseteq) = Q_R(\mathrm{qo}(\mathcal{M}))$.

We verify if $A \in \mathcal{M}$ and $\mathfrak{D}_R(A) \ni x \sqsubseteq x'$, then $\mathfrak{D}_R(A) \ni x'$.

By $\mathfrak{D}_R(A) \ni x$, there exists $v$ such that $R(x, v)$ and $v \subseteq A$. Since $x \sqsubseteq x'$, there exists $v'$ such that $R(x', v')$ and every $x' \in v'$ has $x \in v$ with $x \preceq x'$. Because $x \in A$ and $A \in \mathcal{M}$ is upper-closed with respect to $\preceq$, $x' \in A$. So $v' \subseteq A$. This means $\mathfrak{D}_R(A) \ni x'$.

We are not sure whether $Q_R(\mathrm{qo}(\mathrm{ss}(\mathcal{X})))$ is a WQO for all WQO $\mathcal{X}$, because, according to Lemma 2, the quasi-ordering $\preceq^{\forall}_{\exists}$ is not always a WQO.

Instead, we introduce a stronger set system than an FESS.

**Definition 6** (BESSs)**.** *We say a set system $\mathcal{L} \subseteq P(T)$ is a* better elastic set system *(BESS for short) or $\mathcal{L}$ has* better elasticity*, provided $\mathrm{qo}(\mathcal{L})$ is a BQO.*

**Example 1.** *For every WQO $\mathcal{X}$ which is not a BQO, a set system $\mathrm{ss}(\mathcal{X})$ is an FESS but not a BESS, since $\mathrm{qo}(\cdot)$ is a left-inverse of $\mathrm{ss}(\bullet)$.*

**Lemma 4.**
1. *A quasi-ordering $(X, \preceq)$ is a BQO, if and only if $\mathrm{ss}((X, \preceq))$ is a BESS.*
2. *Every BESS is an FESS.*

*Proof.* (1) By (ii) in the first section. (2) If a set system $\mathcal{L}$ has an infinite learning sequence (see Definition 12) $\langle\langle t_0, L_1\rangle, \langle t_1, L_2\rangle, \ldots\rangle$, then we have a barrier $\mathcal{S}ingl$ and a function $f\ ;\ \{i\} \in \mathcal{S}ingl \mapsto t_i$ such that $t_i \in L_j \not\ni t_j$ hence $t_i \not\preceq_{\mathcal{L}} t_j$ $(i < j)$. This contradicts that $\mathrm{qo}(\mathcal{L})$ is a BQO.

As the class of BQOs is closed under infinitary operations than the class of WQOs is, we prove that the class of BESSs is closed under infinitary operations (i.e., deformations) than the class of FESSs is. See Lemma 1 for the characterization of deformations.

**Theorem 2.** *Assume $\mathcal{L}$ and $\mathcal{M}$ are set systems and $\mathfrak{O} : \mathcal{M} \to \mathcal{L}$ is a deformation. Then if $\mathcal{M}$ is a* BESS*, so is $\mathfrak{O}[\mathcal{M}]$.*

To prove Theorem 2, we have only to verify a following lemma, in view of Corollary 1:

**Lemma 5.** *Let $\mathfrak{O}$ be a deformation from a set system $\mathcal{M}$ to a set system $\mathcal{L}$. If $\alpha$ is a countable infinite indecomposable ordinal, and $\mathrm{qo}(\mathcal{M})$ is an $\alpha\cdot\omega^2$-*WQO*, then $\mathrm{qo}(\mathfrak{O}[\mathcal{M}])$ is an $\alpha$-*WQO*.*

*Proof.* Write $\mathfrak{O}$ as $\mathfrak{O}_R$ for some relation $R \subseteq (\bigcup\mathcal{L}) \times P(\bigcup\mathcal{M})$. Let the quasi-ordering $\mathrm{qo}(\mathcal{M})$ be $(\bigcup\mathcal{M},\ \preceq)$ and $Q_R(\mathrm{qo}(\mathcal{M}))$ be $(\bigcup\mathcal{L},\ \sqsubseteq)$, as in Definition 5. Assume $\mathrm{qo}(\mathfrak{O}[\mathcal{M}])$ is not an $\alpha$-WQO. By Theorem 1 and (ii) in the first section,

$$\mathrm{qo}(\mathfrak{O}[\mathcal{M}]) \supseteq \mathrm{qo}(\mathrm{ss}(Q_R(\mathrm{qo}(\mathcal{M})))) = Q_R(\mathrm{qo}(\mathcal{M})).$$

Since $\mathrm{qo}(\mathfrak{O}[\mathcal{M}])$ is not an $\alpha$-WQO, the quasi-ordering $Q_R(\mathrm{qo}(\mathcal{M})) = (\bigcup\mathcal{L},\ \sqsubseteq)$ is neither. By Proposition 4, there are smooth barrier $B$ of $\mathrm{o.t.}(B) \leq \alpha$ and a function $f : B \to \bigcup\mathfrak{O}[\mathcal{M}]$ such that for all $u, v \in B$, $u \triangleleft v$ implies $f(u) \not\sqsubseteq f(v)$. By Proposition 5 (1), for all $t \in B^2$, we have $\pi_0(t), \pi_1(t) \in B$ and $\pi_0(t) \triangleleft \pi_1(t)$. So $f(\pi_0(t)) \not\sqsubseteq f((\pi_1(t)))$ for all $t \in B^2$. By Definition 5, for some $v$ with $R(f(\pi_0(t)),\ v)$ and for all $v'$, if $R(f(\pi_1(t)),\ v')$ then $v \not\preceq^{\forall}_{\exists} v'$. For each $t$, let $g(t)$ be one of such $v$. Then $g$ is a function from $B^2$ to $P(\bigcup\mathcal{M})$.

Then for all $t, t \in B^2$, we have $g(t) \not\preceq^{\forall}_{\exists} g(t')$ whenever $t \triangleleft t'$. In other words, the function $g$ is bad with respect to the powerset ordering $\preceq^{\forall}_{\exists}$. To verify it, first recall $\pi_1(t) = \pi_0(t')$ by Proposition 5 (2). By the definition of $g$, we have $g(t) \not\preceq^{\forall}_{\exists} v'$ whenever $R(f(\pi_1(t)),\ v')$. Moreover $R(f(\pi_0(t')),\ g(t'))$. Because Proposition 5 (2) implies $\pi_1(t) = \pi_0(t')$, we have $R(f(\pi_1(t)),\ g(t'))$. Therefore, $g(t) \not\preceq^{\forall}_{\exists} g(t')$.

By Proposition 3 and Proposition 5 (4), $\mathrm{o.t.}(B^2) = \mathrm{o.t.}(B)\cdot\omega \leq \alpha\cdot\omega$. Because $B^2$ is a barrier by Proposition 5 (3), $(P(\bigcup\mathcal{L}),\ \preceq^{\forall}_{\exists})$ is not an $(\alpha\cdot\omega)$-WQO, which contradicts Proposition 6.

A following immediate corollary of Theorem 2 may be useful in developing computational learning theory of $\omega$-languages [37]: Let $\Sigma^\infty$ be a set of possibly infinite sequences of elements in an alphabet $\Sigma$, and $\mathcal{L}$ be a set system over $\Sigma^\infty$. The concatenation operation of two sequences is defined similarly as that of two finite sequences except that for an infinite sequence $u$ and sequence $v$, the concatenation $uv$ is defined as $u$. For $L \subseteq \Sigma^\infty$, the *$\omega$-closure* [37] $L^\omega$ of $L$ is the set of infinitely iterated concatenation $u_1u_2u_3\cdots$ of sequences $u_1, u_2, \ldots \in L$.

**Corollary 2** (Closure of $\omega$-languages)**.** *If $\mathcal{L}$ is a* BESS*, so are following classes:*

(1) $\{L^\omega\ ;\ L \in \mathcal{L}\}$.
(2) $\{L^{\mathrm{sh}}\ ;\ L \in \mathcal{L}\}$. *Here $L^{\mathrm{sh}}$ is the shuffle-closure $\{\varepsilon\} \cup L \cup (L \diamond L) \cup ((L \diamond L) \diamond L) \cup \cdots$, and $L \diamond L'$ is the set of $u_1v_1u_2v_2\cdots$ such that $u_1u_2\cdots \in L$, $v_1v_2\cdots \in L'$ and $u_i, v_i \in \Sigma^*$ $(i \geq 1)$.*

## 4. Linearizations of set systems and powerset orderings

Many study on WQOs use de Jongh-Parikh's theorem [13]: "The order-type $\mathrm{otp}(\mathcal{X})$ of a WQO $\mathcal{X}$ is the *maximum* of order-types of the linearizations of $\mathcal{X}$."

We wish to require that if a linear order $\mathcal{Y}$ is a linearization of a quasi-ordering of $\mathcal{X}$, then $\mathrm{ss}\,(\mathcal{Y})$ is a 'linearization' of a $\mathrm{ss}\,(\mathcal{X})$. Then a 'linearization' of a set system $\mathcal{L} \subseteq P(T)$ should be a set system $\mathcal{M} \subseteq P(T)$ linearly ordered by set inclusion and $\mathcal{M}$ consists of unions of members of $\mathcal{L}$ such that for any $L \in \mathcal{L}$ there exists a subfamily $\mathcal{L}' \subseteq \mathcal{L}$ such that $L \in \mathcal{L}'$ and $\bigcup \mathcal{L}' \in \mathcal{M}$.

**Lemma 6.** *For any* FESS $\mathcal{L}$, *if* $\mathrm{qo}\,(\mathcal{L})$ *is a* WQO *then the class* $\hat{\mathcal{L}} := \{ \bigcup \mathcal{M} \ ; \ \emptyset \neq \mathcal{M} \subseteq \mathcal{L} \}$ *containing* $\mathcal{L}$ *closed under arbitrary unions is an* FESS.

*Proof.* Since each member $\bigcup \mathcal{M} \in \hat{\mathcal{L}}$ is upper-closed with respect to the WQO $\mathrm{qo}\,(\mathcal{L})$, any learning sequences (see Definition 12) of the set system $\hat{\mathcal{L}}$ are those of the set system $\mathrm{ss}\,(\mathrm{qo}\,(\mathcal{L}))$. By the premise $\mathrm{qo}\,(\mathcal{L})$ is a WQO, so $\mathrm{ss}\,(\mathrm{qo}\,(\mathcal{L}))$ is an FESS. Therefore any learning sequence should be finite. Hence $\hat{\mathcal{L}}$ is an FESS.

**Conjecture 1.** *Define a suitable linearization of a set system. Do we have*

$$\dim \mathcal{X} = \max \{\dim \mathcal{Y} \ ; \ \mathcal{Y} \textit{ is a linearization of } \mathcal{X}\}$$

*for any set system* $\mathcal{X}$ *?*

We characterize the set systems $\mathcal{L}$ such that the set system of arbitrary (finite) unions of members of $\mathcal{L}$ is an FESS, from viewpoint of WQOs, BQOs, and the powerset ordering $\preceq^{\forall}_{\exists}$.

**Theorem 3.** *For any set system* $\mathcal{L}$, *the following are equivalent:*

1. *the quasi-ordering* $\mathrm{qo}\,(\mathcal{L}) = (\bigcup \mathcal{L}, \leq)$ *satisfies the condition* (1) *of Lemma 2. Namely,*
   - (a) $\mathrm{qo}\,(\mathcal{L})$ *is a* WQO*; and*
   - (b) *Let* $F$ *be any function from* $[\omega]^2$ *to* $\bigcup \mathcal{L}$. *Then if* $F(\{i,j\}) < F(\{i,j+1\})$ *for any* $i < j < \omega$, *then there are* $i < j < k < \omega$ *such that* $F(\{i,j\}) < F(\{j,k\})$.
2. $\mathcal{L}^{<\omega}$ *is an* FESS.
3. $\hat{\mathcal{L}}$ *is an* FESS.

*Proof.* Write $\mathrm{qo}\,(\mathcal{L}) = (\bigcup \mathcal{L}, \preceq)$. Assume the condition (2) is false. Then there are an infinite sequence of $\mathcal{M}_n \subseteq \mathcal{L}^{<\omega}$ $(n = 1, 2, \ldots)$ and an infinite sequence $x_n \in \bigcup \mathcal{L}$ $(n = 1, 2, \ldots)$ such that

$$v_n := \{x_1, \ldots, x_{n-1}\} \subseteq \bigcup \mathcal{M}_n \not\ni x_n. \tag{4}$$

If there are $n < m$ such that $v_n \preceq^{\forall}_{\exists} v_m$, then $x_i \preceq x_{m-1}$ for some $i < n$. However, (4) implies $x_i \in \bigcup \mathcal{M}_{m-1} \not\ni x_{m-1}$. By the definition of $\preceq$, $x_i \preceq x_{m-1}$ implies $x_i \in \bigcup \mathcal{M}_{m-1} \ni x_{m-1}$. A contradiction. Thus the powerset ordering $(P(\bigcup \mathcal{L}), \preceq^{\forall}_{\exists})$ is not a WQO. By Lemma 2, the condition (1) is false.

Conversely, assume the condition (1) is false. Since Lemma 2 implies that the condition (1) is equivalent to the well-quasi-orderedness of the powerset ordering $\preceq^{\forall}_{\exists}$, we have an infinite sequence $(v_i)_i$ such that for all $i < j$, $v_i \not\preceq^{\forall}_{\exists} v_j$ but $v_i \subseteq \bigcup \mathcal{L}$. Then there is $y^j \in v_j$ such that for all $y^i \in v_i$ we have $y^i \not\preceq y^j$. By the definition of $\preceq$, there is a sequence $(L_{i,j})_{i<j}$ such that $y^i \in L_{i,j} \not\ni y^j$. Hence the sequence

$\left(\bigcup_{i<j} L_{i,j}\right)_j$ is an infinite learning sequence in $\mathcal{L}^{<\omega}$. So $\mathcal{L}^{<\omega}$ is not an FESS. Thus the condition (3) is false.

The equivalence between the condition (1) and the condition (2) can be similarly proved.

**Example 2.** *By* [30][40]*, we have an* FESS

$$\mathcal{L}_1 := \{\{i\} \cup \{k\,;\ k \geq j\}\ ;\ i, j \in \omega\},$$

*because it is the memberwise union of an* FESS $\{\ \{k \in \omega\ ;\ k \geq j\}\ ;\ j \in \omega\ \}$ *and an* FESS $\mathcal{S}ingl$*. But* $(\mathcal{L}_1)^{<\omega}$ *is not an* FESS *according to* [10, Proposition 2.1.27]*. The last assertion is an easy corollary of Theorem 3, because* $\mathrm{qo}(\mathcal{L}_1) = (\omega, =)$ *and is not a* WQO*.*

**Corollary 3.** *If* $\mathcal{L}$ *is a* BESS*, then both of* $\mathcal{L}^{<\omega}$ *and* $\hat{\mathcal{L}}$ *are* FESS*s.*

*Proof.* As $\mathcal{L}$ is a BESS, the quasi-ordering $\mathrm{qo}(\mathcal{L})$ is a BQO by the definition, and hence is an $\omega^2$-WQO, by the definition of BQO. By Lemma 2, we have the condition (1) of Theorem 3 and thus the desired conclusions.

As the class of BQOs enjoys the closure properties with respect to possibly infinitary constructions, we conjecture a following:

**Conjecture 2.** *If* $\mathcal{L}$ *is a* BESS*, then both of* $\mathcal{L}^{<\omega}$ *and* $\hat{\mathcal{L}}$ *are* BESS*s.*

We contrast our characterization of set systems $\mathcal{L}$ having an FESS $\mathcal{L}^{<\omega}$, with Shinohara-Arimura's sufficient condition [39] for a set system $\mathcal{L}$ to have an FESS $\mathcal{L}^{<\omega}$.

**Definition 7.** *Let* $\mathcal{L}$ *be a set system over* $X$*.*

$\mathcal{L}$ *is said to have a* finite thickness (FT)*, provided that for any* $x \in X$ $\#\{L \in \mathcal{L}\ ;\ x \in L\} < \infty$*.* $\mathcal{L}$ *is said to have* no-infinite-antichain property (NIA)*, provided that* $\mathcal{L}$ *has no infinite antichain with respect to the set-inclusion* $\subseteq$*.*

The set system $\mathcal{S}ingl$ has an FT but not NIA. If a set system has an FT, then it is an FESS [39].

**Proposition 7** ([39])**.** *If* $\mathcal{L}$ *has an* FT *and* NIA*, then* $\mathcal{L}^{<\omega}$ *is an* FESS*.*

However the conjunction of FT and NIA is not preserved by the operation $(\cdot)^{<\omega}$.

**Lemma 7.** (1) *A set system* $\mathcal{L}_2 = \{\ [i,\ \infty) \cap \mathbb{Z}\ ;\ i \geq 1\} \cup \{\{0\}\}$ *has an* FT *and* NIA *but* $\mathcal{L}_3 := (\mathcal{L}_2)^{<\omega}$ *is an* FESS *without an* FT *(*[9]*).*

(2) *The converse of Proposition 7 is false. Actually,* $(\mathcal{L}_3)^{<\omega}$ *is an* FESS *but* $\mathcal{L}_3$ *does not have an* FT*.*

**Lemma 8.** (1) *For any* FESS $\mathcal{L}$*,* $\mathcal{L}$ *has* NIA *if and only if* $(\mathcal{L}, \supseteq)$ *is a* WQO*.*

(2) *If* $\mathcal{L}^{<\omega}$ *is an* FESS*, then* $\mathcal{L}^{<\omega}$ *has* NIA*.*

*Proof.* (1) The if-part is immediate from the definition of WQOs. Assume there is an infinite descending chain $(L_i)_i \subseteq \mathcal{L}$ with respect to $\supseteq$. Hence, $L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \cdots$. By putting $x_i \in L_{i+1} \setminus L_i$, we have $\langle\langle x_1, L_2\rangle, \langle x_2, L_3\rangle, \cdots\rangle$ is an infinite learning sequence. This contradicts that $\mathcal{L}$ is an FESS.

(2) Suppose $(\bigcup \mathcal{M}_n)_n$ $(\mathcal{M}_n \subseteq \mathcal{L}\ ;\ n = 1, 2, \ldots)$ is an infinite antichain in $\mathcal{L}^{<\omega}$. Then $\left(\bigcup_{n<m} \bigcup \mathcal{M}_n\right)_m$ is a strictly ascending chain in $\mathcal{L}^{<\omega}$. But $\mathcal{L}$ is an FESS.

Following relation holds among (continuous) deformations, NIA and FT:

**Lemma 9.** (1) *If a set system $\mathcal{L}$ has* NIA*, so does $\mathfrak{D}[\mathcal{L}]$ of $\mathcal{L}$ for any deformation $\mathfrak{D}$.*

(2) *For any nonempty set $X$ and for any $x \in X$, a function $\mathfrak{D} : P(X) \to P(X)$ ; $A \mapsto A \cup \{x\}$ is a continuous deformation. Thus even if $\mathcal{L}$ has an* FT*, $\mathfrak{D}[\mathcal{L}]$ does not.*

*Proof.* (1) If the image $\{\mathfrak{D}(L_i)\}_i$ of $\{L_i\}_i \subseteq P(T)$ by a deformation $\mathfrak{D}$ is an infinite antichain, then, for any distinct $i, j \in \omega$ there exists $n_{i,j} \in \mathfrak{D}(L_i) \setminus \mathfrak{D}(L_j)$. Let $\mathfrak{D}$ be as in the equation (1). Then we have $\exists v_{i,j} \in P(L_i)$. $R(n_{i,j},\, v_{i,j})$ and $\forall v \in P(L_j)$. $\neg R(n_{i,j}, v)$. Therefore $v_{i,j}$ is not a subset of $L_j$. Thus, there exists $a_{i,j} \in v_{i,j} \setminus L_j \ \subseteq\ L_i \setminus L_j$. Hence, $\{L_i\}_i$ is also an infinite antichain. (2) It is immediate.

Finally, we remark that a condition for a set system $\mathcal{L}$ to satisfy $\mathcal{L}^{<\omega}$ being an FESS does not depend on the structure of $\mathcal{L}$ with respect to the set-inclusion, in view of the assertion (2) and the assertion (5) of following Lemma 10.

We recall that a quasi-ordering $\mathcal{X} = (X, \preceq)$ is a WQO, if and only if any upper-closed subset of $X$ is a finite union of principal filters [17].

**Definition 8.** *For a quasi-ordering $\mathcal{X} = (X, \preceq)$, define $|\mathrm{PF}\,(\mathcal{X})|$ to be the set of principal filters of $\mathcal{X}$. Let $\mathrm{PF}\,(\mathcal{X})$ be $|\mathrm{PF}\,(\mathcal{X})|$ ordered by the reverse set-inclusion.*

**Fact 1.** *Let $\mathcal{X}$ be a quasi-ordering.*

(1) $\mathrm{ss}\,(\mathcal{X}) = |\mathrm{PF}\,(\mathcal{X})|^{<\omega}$ *if and only if $\mathcal{X}$ is a* WQO*.*

(2) $\mathcal{X} = \mathrm{qo}\,(\ |\mathrm{PF}\,(\mathcal{X})|\ )$.

(3) *$\mathcal{X}$ is order-isomorphic to $\mathrm{PF}\,(\mathcal{X})$ for any partial ordering $\mathcal{X}$.*

(4) $\dim |\mathrm{PF}\,(\mathcal{X})| \ \leq\ \dim \mathrm{ss}\,(\mathcal{X}) = \mathrm{otp}(\mathcal{X})$.

(5) *For the partial order*

$$\mathcal{X} = (\{b\} \cup \{a_i\ ;\ i \in \omega\},\ \{(b,\ a_i)\ ;\ i \in \omega\})\,,$$

*we have* $\dim \mathrm{PF}\,(\mathcal{X}) = 1$ *but* $\dim \mathcal{X} = \infty$.

**Lemma 10.** (1) *$\mathcal{L}_1$ has* NIA*.*

(2) *A quasi-ordering $(\mathcal{L}_1,\ \supseteq)$ is order-isomorphic to $\mathrm{PF}\,(\ (\mathcal{L}_1,\ \supseteq)\ )$. They are* WQO*s.*

(3) *None of $(\mathcal{L}_1,\ \supseteq)$ and $\mathrm{PF}\,(\ (\mathcal{L}_1,\ \supseteq)\ )$ is a* BQO*.*

(4) *None of $\mathcal{L}_1$ and $|\mathrm{PF}\,(\ (\mathcal{L}_1,\ \supseteq)\ )|$ is a* BESS*.*

(5) *$(\mathcal{L}_1\,)^{<\omega}$ is not an* FESS*, but $|\mathrm{PF}\,(\ (\mathcal{L}_1,\ \supseteq)\ )|^{<\omega}$ is.*

*Proof.* (1) By [11, Proposition 3.3]).

(2) By Lemma 8 (1) and the assertion (1) of this Lemma, the partial ordering $(\mathcal{L}_1, \supseteq)$ is a WQO.

(3) It is because a following function $F$ does not satisfy Lemma 2 (1) (b):

$$F(\{i,j\}) := \{i\} \cup \{k\ ;\ k > j\}. \quad (i < j)$$

(4) Let $(\omega, \preceq)$ be $\mathrm{qo}\,(\mathcal{L}_1)$. Then $n \not\preceq m$ $(n < m)$ because $n \in F(\{n,m\}) \not\ni m$, while $n \not\preceq m$ $(n > m)$ because $n \in F(\{n, n+1\}) \not\ni m$. Therefore $\mathrm{qo}\,(\mathcal{L}_1)$ is not a WQO, hence is not a BQO. So $\mathcal{L}_1$ does not satisfy the condition 1 of Theorem 3. Hence $(\mathcal{L}_1\,)^{<\omega}$ is not an FESS.

By Fact 1 (2) and the assertion (2) of this Lemma, $\mathrm{qo}\,(|\mathrm{PF}\,(\ (\mathcal{L}_1,\ \supseteq)\ )|)$ is $(\mathcal{L}_1,\ \supseteq)$ which is not a BQO by the assertion 3 of this Lemma.

(5) Since $\mathrm{qo}\,(\mathcal{L}_1)$ is not a WQO by the proof of the assertion (4) of this lemma, $(\mathcal{L}_1)^{<\omega}$ is not an FESS because of Theorem 3. By Fact 1 (1), the set system $|\mathrm{PF}\,((\mathcal{L}_1, \supseteq))|^{<\omega}$ is $\mathrm{ss}\,((\mathcal{L}_1, \supseteq))$, which is an FESS by the assertion (2) of this Lemma.

Although $|\mathrm{PF}\,((\mathcal{L}_1, \supseteq))|$ is not a BESS, it satisfies the condition 1 of Theorem 3, according to the assertion (5) of Lemma 10.

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## Appendix A. Ramsey's numbers for well-founded trees and order type of set systems

Let $\mathcal{X}_i$ be a quasi-ordering with the maximal order type $\mathrm{otp}(\mathcal{X}_i) < \omega$ and $\mathcal{L}_i$ be a set system with the order type $\dim \mathcal{L}_i < \omega$ $(i = 1, 2)$. Let $\mathrm{Ra}(n, m)$ be the Ramsey number [16] of $n$ and $m$. Then we prove

(1) [3, Lemma 6] For the memberwise union $\mathcal{L}_1 \,\widetilde{\cup}\, \mathcal{L}_2 = \{L_1 \cup L_2 \ ; \ L_1 \in \mathcal{L}_1,\ L_2 \in L_2\}$,
$$\dim(\mathcal{L}_1 \,\widetilde{\cup}\, \mathcal{L}_2) + 1 < \mathrm{Ra}(\dim \mathcal{L}_1 + 2, \dim \mathcal{L}_2 + 2).$$
(2) [3, Theorem 8] $\mathrm{otp}\,(\mathcal{X}_1 \cap \mathcal{X}_2) < \mathrm{Ra}(\mathrm{otp}(\mathcal{X}_1) + 1,\ \mathrm{otp}(\mathcal{X}_2) + 1)$.

We wish to generalize these two for the case $\dim \mathcal{L}_i$ $(i = 1, 2)$ being general ordinal numbers. To directly generalize the proof argument of the two, we pose a following question. By a tree, we mean a prefix-closed set of possibly infinite sequences. A well-founded tree is, by definition, a tree with all the elements being finite sequences.

**Conjecture 3.** *Is there a reasonably simple, ordinal binary (partial) function $F$ on ordinal numbers such that "for all ordinal numbers $\beta$ and $\gamma$ there exists an ordinal number $\alpha \le F(\beta, \gamma)$ with a following property: for any coloring of any well-founded tree $T_0$ of order type $\alpha$ with red and black, either there is a well-founded tree $T_1$ of order type $\beta$ such that $T_1$ is homeomorphically embedded into the red nodes of $T_0$, or there is a well-founded tree $T_2$ of order type $\gamma$ such that $T_2$ is homeomorphically embedded into the black nodes of $T_0$."*

## Appendix B. Initial segments of quasi-ordering : computability theoretic view

For every nonnegative integer $z$, a set $\{z_1, \ldots, z_m\}$ of nonnegative integers $z_1, \ldots, z_m$ with $z = 2^{z_1} + \cdots + 2^{z_m}$ is denoted by $E_z$.

**Definition 9.** *A set $A \subseteq \omega$ is called* positively reducible *via a recursive function $f : \omega \to \omega$ to $B \subseteq \omega$ ($A \le_p B$ via $f$, in symbol), provided that for all $x$, $x \in A$ if and only if there exists $y \in E_{f(x)}$ such that $E_y \subseteq B$. Intuitively, a finite set $E_y$ means a conjunction of Boolean variables, and a finite set $E_{f(x)}$ means a disjunction of such conjunctions $E_y$ over $y \in E_{f(x)}$. We write $A \le_p B$ if there exists a recursive function $f : \omega \to \omega$ such that $A \le_p B$ via $f$.*

We observe that for any recursive relation $R \subseteq \omega \times [\omega]^{<\omega}$ and for any class $\mathcal{L} \subseteq P(\omega)$, the image $\mathfrak{O}_R[\mathcal{L}]$ is the class of sets *positively reducible* [20] to some sets in $\mathcal{L}$ "uniformly" via a single recursive function
$$f_R(x) = \sum_{R(x,v)} 2^{\sum_{i \in v} 2^i}.$$

According to [20], the class of *semirecursive sets* is closed by the positive reduction (equivalent to effective continuous deformation, in spirit), and a semirecursive set is exactly an *initial segment* of some recursive linear ordering on $\omega$.

**Definition 10.** *A set $M \subseteq \omega$ is called* semirecursive [20] *if there is a recursive function $\psi$ of two variables such that*
$$\begin{aligned}&(x \in M \wedge y \notin M) \vee (x \notin M \wedge y \in M)\\ &\Longrightarrow \psi(x, y) \in \{x, y\} \cap M.\end{aligned} \tag{5}$$
*In* [21]*, Jockusch and Owings introduced a following generalization of a semirecursive set: $M \subseteq \omega$ is* semi-r.e. *if and only if there exists a partial recursive function $\psi$ of two variables such that for all $x, y \in \omega$*
$$\Big(x \in M \vee y \in M \implies \psi(x, y) \in \{x, y\} \cap M\Big).$$

*Furthermore, they introduced a following generalization of a semi-r.e. set: $M$ is* weakly semirecursive *if and only if there exists a partial recursive function $\psi$ of two variables such that the condition* (5) *holds.*

*The (partial) function $\psi$ is called a* selector function *of the semirecurisve (semi-r.e., weakly semirecursive) set $M$.*

We adapt the notion of the initial segments of partial orderings [26, p. 136], as follows:

**Definition 11.** *For any quasi-ordering $\preceq$ on $\omega$, we say $M \subseteq \omega$ is an* initial segment *of $\preceq$, if and only if any of $M$ is strictly smaller with respect to $\preceq$ than any of the complement $\overline{M}$.*

Every initial segment of a quasi-ordering is trivial, if and only if the undirected graph induced by the quasi-ordering is not connected. A non-trivial initial segment may have downward branching.

We characterize a weakly semirecursive sets and semi-r.e. sets by initial segments of quasi-orderings.

**Theorem 4.** *A set $M$ is weakly semirecursive if and only if $M$ is an initial segment of an r.e. quasi-ordering.*

*Proof.* ($\Rightarrow$) By [26, Theorem 4.1]. ($\Leftarrow$) Let the witnessing quasi-ordering be $\leq$. Put

$$\psi(x,y) := \begin{cases} x, & (x \leq y \text{ and } x \neq y); \\ y, & (y \leq x \text{ and } x \neq y); \\ \uparrow, & \text{otherwise.} \end{cases} \tag{6}$$

Then $\psi$ is clearly a partial recursive function. Assume $x \in M \not\ni y$. Because $M$ is an initial segment of $\leq$ in a sense of Definition 11, we have $x \leq y$ and $x \neq y$. By the definition of $\psi$, we have $\psi(x,y) = x$. On the other hand, assume $x \notin M \ni y$. Then $y \leq x$ and $x \neq y$. To sum up, $\psi(x,y) \in \{x,y\} \cap M$. Thus $M$ is a weakly semirecursive set with $\psi$ being a selector function.

We can prove a similar result for semi-r.e. sets.

**Theorem 5.** *A set $M$ is semi-r.e. if and only if $M$ is a linearly ordered initial segment of an r.e. quasi-ordering.*

*Proof.* Only if-part is by [26, Theorem 5.1]. To prove the converse, assume $x \in M$ without loss of generality. When $y \in M$, we have $x \leq y$ or $y \leq x$ because $M$ is linearly ordered. By (6), we have $\psi(x,y) \in \{x,y\} \cap M$. When $y \notin M$, $x \leq y$ and $x \neq y$ because $M$ is an initial segment of $\leq$. By (6), we have $\psi(x,y) = x$.

A lemma similar to "If $A \leq_p B$ and $B$ is semirecursive, then $A$ is semirecursive" [20, Theorem 4.2] holds for semi-r.e. sets and weakly semirecursive sets.

**Lemma 11.** *If $A \leq_p M$ and $M$ is semi-r.e. (weakly semirecursive, resp.), then so is $A$.*

*Proof.* Let $\psi$ be a selector function of $M$. Because $A \leq_p M$, the set $A$ is many-one reducible to $M$, by [20, Theorem 4.2 (ii)]. So there exists a recursive function $g$ such that

$$x \in A \iff g(x) \in M. \tag{7}$$

Define a partial recursive function $\psi'$ by

$$\psi'(x,y) = \begin{cases} x, & (\psi(g(x),\, g(y)) = g(x)); \\ y, & (\psi(g(x),\, g(y)) = g(y)); \\ \uparrow, & \text{(otherwise)}. \end{cases} \tag{8}$$

(i) Assume $M$ is weakly semirecursive. Suppose $x \in A \not\ni y$ without loss of generality. By (7), $g(x) \in M \not\ni g(y)$. Thus $\psi(g(x), g(y)) \in \{g(x), g(y)\} \cap M$. By $g(y) \notin M$, we have $\psi(g(x),\, g(y)) = g(x) \in M$. Hence $\psi'(x,\, y) = x \in \{x, y\} \cap A$. Therefore $A$ is weakly semirecursive with a selector function $\psi'$.

(ii) Assume $M$ is semi-r.e. Suppose $x \in A$ or $y \in A$. Then $g(x) \in M$ or $g(y) \in M$. So $\psi(g(x),\, g(y)) \in \{g(x),\, g(y)\} \cap M$. When $\psi(g(x),\, g(y)) = g(x)$, by a similar argument of (i), we have $\psi'(x, y) \in \{x, y\} \cap A$. When $\psi(g(x),\, g(y)) = g(y)$, $\psi'(x, y) = y \in \{x, y\} \cap A$. Thus $A$ is semi-r.e. with a selector function $\psi'$.

**Corollary 4.** *Let $R \subseteq \omega \times [\omega]^{<\omega}$ be a recursive relation. Then if $A$ is semirecursive (semi-r.e., weakly semirecursive resp.), then so is $B \subseteq \omega$ where $B = \mathfrak{D}_R(A)$.*

## Appendix C. A new order type of a set system

**Definition 12.** *A* learning sequence *of a set system $\mathcal{L} \subseteq P(T)$ is, by definition, a possibly infinite sequence*

$$\Big\langle \langle t_0, A_1\rangle, \langle t_1, A_2\rangle, \dots \Big(, \langle t_n, A_{n+1}\rangle\Big) \Big\rangle$$

*such that for each $i < n$ $\{t_0, \dots, t_i\} \subseteq A_{i+1} \in \mathcal{L}$. In particular, we call the sequence* bad *if $A_{i+1} \not\ni t_{i+1}$ for each $i$.*

*We say a set system $\mathcal{L} \subseteq P(T)$ has* infinite elasticity*, provided that there are infinite bad learning sequences. Otherwise, we say $\mathcal{L}$ has an* FE*, and call $\mathcal{L}$ an* FESS.

*Let $T$ be a well-founded tree. For each node $\sigma$ of $T$, let the ordinal number $|\sigma|$ be the supremum of $|\sigma'| + 1$ such that $\sigma' \in T$ is an immediate extension of $\sigma$. Then the* order type *$|T|$ of the well-founded tree $T$ is defined by the ordinal number $|\langle\,\rangle|$ assigned to the root $\langle\,\rangle$ of $T$. For a tree $T$ which is not well-founded, let $|T|$ be $\infty$.*

*The order type of $\mathcal{L}$, denoted by $\dim \mathcal{L}$, is, by definition, the order type of the tree of bad learning sequences of $\mathcal{L}$.*

In the premise of Proposition 1, we cannot replace the domain of the continuous function $\mathfrak{D} : \{0,1\}^{\bigcup \mathcal{M}} \to \mathcal{L}$ with a set system $\mathcal{M}$. We have following counterexample: $\mathcal{M} = \{\ \{i\}\ ;\ i \in \omega\}$ is a discrete subspace of the product topology $\{0,1\}^{\omega}$ and hence any function from the relative topology $\mathcal{M}$ to a set system $\mathcal{L}$ is continuous even if $\mathcal{L}$ is not an FESS.

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